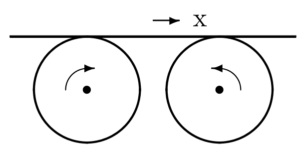
Moretti 4/30/2002

Demonstration 1: A foot-long horizontal steel rod rests on top of counter-rotating pulleys of 2.25 inch effective radius.[1] The horizontal axes of the pulleys are spaced six inches apart. The tops of the pulleys normally rotate towards each other. The grooves in the pulleys have sides angled at 45 degrees.



Heuristic approach: initially investigate transverse velocities of the rod which are less than the inward velocity $R\dot{\theta}$ of each pulley surface (*i.e.,* fast-turning pulleys)

$$\left|\dot{x}\right| < \left|R\dot{\theta}\right|$$

so that rod does not experience static-friction lock-on to either pulley. (We are not studying the case where the rod and pulleys stay locked-together, and oscillate against the elasticity of the figure-eight belt connecting the pulleys.)

For a **first approximation**, take Coulomb friction coefficient μ =constant (independent of relative velocity): friction force is proportional to contact force due to weight loading. Since the pulley sides support the rod at a 45-degree angle, the total contact force at each pulley is increased by a factor of $\sqrt{2}$ over the load.

- When rod is centered ($x = 0$), each pulley supports a load of $W/2$ (half of the rod's weight W) and provides a centering force component of $\sqrt{2}\mu W/2$; since they are counter-rotating, the net force is $F = 0$.
- As the rod approaches the maximum stable displacement $\pm\ell$ (equal to one-half of the span $2\ell = 0.5$ ft), the force approaches $\mp\sqrt{2}\mu W$.
- In between, the relative loads are inversely related to the relative distances of the support-points from the center of gravity, resulting in a net friction force

$$F = \sqrt{2}\mu\left(\frac{\ell - x}{2\ell}\right)W - \sqrt{2}\mu\left(\frac{\ell + x}{2\ell}\right)W = -\sqrt{2}\mu\left(\frac{x}{\ell}\right)W$$

This is a linear centering force (unless the pulleys turn the wrong way, in which case the sign changes). Adding the inertial forces due to the rod's mass W/g,

[1] For teachers, Endevco (http://www.endevco.com/resources/uniforum.php) has provided a demonstration model constructed from a design by D.S. Stutts (U. Missouri–Rolla), inspired by Problem #2-6 in *Vibration of Mechanical and Structural Systems: with Microcomputer Applications,* 2nd Edition, edited by J. Lenchek, Harper-Collins, 1994, ISBN: 0060432616.

the normalized Governing Equation with this restoring force is

$$\ddot{x} + \left(\frac{\sqrt{2}\mu g}{\ell} \right) x = 0 \tag{1a.}$$

which is a linear equation with a vibrational response to initial conditions (unless the pulleys rotation reverses and the sign changes, in which case the solution diverges). By measuring the period $\tau_n = 2\pi\sqrt{\ell/\sqrt{2}\mu g}$ of the system (approximately 1 second), we can obtain the friction coefficient[2]

$$\mu = \left(\frac{2\pi}{\tau_n}\right)^2 \frac{\ell}{\sqrt{2}g} = \left(\frac{2\pi}{1\text{ sec}}\right)^2 \frac{0.25\text{ ft}}{} \left| \frac{}{\sqrt{2}} \right| \frac{\text{sec}^2}{32\text{ ft}} = 0.2 = 20\%$$

If one has other rods (*e.g.,* brass, aluminum, wood, or plastic), one can compare coefficients of friction.

There is a geometrical simplification in this analysis: actually, the center of gravity is not in the plane of the contact points, but $r/\sqrt{2}$ above it. Hence there is an additional weight transfer due to the acceleration. The corrected governing equation for the central range of motion is

$$\frac{W}{g}\ddot{x} = \sqrt{2}\mu\left(\left(\frac{\ell - x}{2\ell}\right)W + \frac{(r/\sqrt{2})}{(2\ell)}\frac{W}{g}\ddot{x}\right) - \sqrt{2}\mu\left(\left(\frac{\ell + x}{2\ell}\right)W - \frac{(r/\sqrt{2})}{(2\ell)}\frac{W}{g}\ddot{x}\right)$$

which simplifies to

$$\ddot{x} + \left(\frac{\sqrt{2}\mu g}{(\ell - \mu r)} \right) x = 0 \tag{1b.}$$

Since our value of $\mu r \approx 0.05$ inch is only about 2% of $\ell = 3$ inches, the frequency adjustment is only about 1% and may generally be neglected. The geometrical analysis needs to be refined only for extreme amplitudes, when the vertical offset momentarily causes the rod to lift off from one pulley, and the support point shifts around the perimeter of the other pulley.

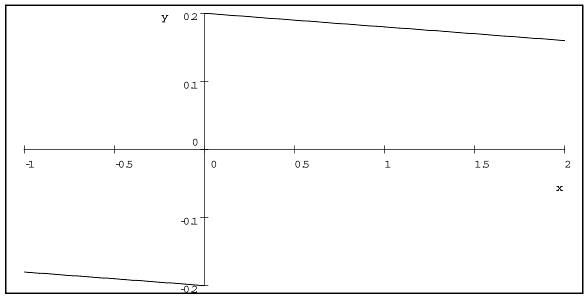
On the other hand, realistic friction is much more complex than a constant value for μ. For a **second approximation**, consider friction expressions such as $\mu\left(1 \pm \epsilon \left|\dot{x}_{\text{rel}}\right|\right)$ which vary slightly with speed: a positive sign on ϵ represents the trend for an oiled surface; while a *negative* sign is typical for metal-to-metal

[2] An alternative way of checking friction coefficient is to hold the pulleys stationary, tilt the assembly, and record the angle at which sliding is just maintained

$$\begin{aligned} \sqrt{2}\mu mg\cos\theta &= mg\sin\theta \\ \sqrt{2}\mu &= \tan\theta \end{aligned}$$

This is effective for obtaining the initial break-loose μ_{max}, but hard to execute for steadily sliding μ.

dry friction and the applicable range is limited to $\left|\dot{x}_{\mathrm{rel}}\right| < 1/\epsilon$:



Note that ϵ is not dimensionless, but has inverse-velocity units of sec/ft or s/m. The relative velocity at each pulley (taken as positive if it is towards the right-hand pulley)

$$\begin{aligned}\dot{x}_{\mathrm{rel}} &= \dot{x} - \left|R\dot{\theta}\right| \text{ on the left-hand pulley} \\ \dot{x}_{\mathrm{rel}} &= \dot{x} + \left|R\dot{\theta}\right| \text{ on the right-hand pulley}\end{aligned}$$

Since our initial investigation is restricted to small velocities $\left|\dot{x}\right| < \left|R\dot{\theta}\right|$

$$\begin{aligned}\left|\dot{x}_{\mathrm{rel}}\right| &= \left|R\dot{\theta}\right| - \dot{x} \text{ on the left-hand pulley} \\ \left|\dot{x}_{\mathrm{rel}}\right| &= \left|R\dot{\theta}\right| + \dot{x} \text{ on the right-hand pulley}\end{aligned}$$

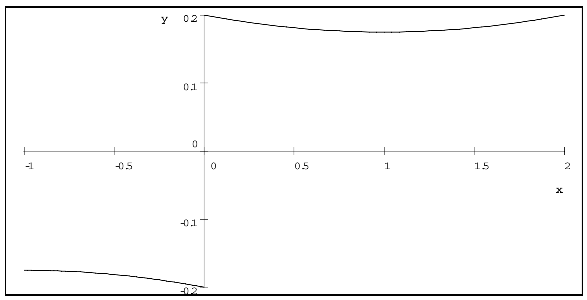
$$\begin{aligned}F &= \sqrt{2}\mu\left[1 \pm \epsilon\left(\left|R\dot{\theta}\right| - \dot{x}\right)\right]\left(\frac{\ell - x}{2\ell}\right)W - \sqrt{2}\mu\left[1 \pm \epsilon\left(\left|R\dot{\theta}\right| + \dot{x}\right)\right]\left(\frac{\ell + x}{2\ell}\right)W \\ &= -\sqrt{2}\mu\left[1 \pm \epsilon\left|R\dot{\theta}\right|\right]\left(\frac{x}{\ell}\right)W - \sqrt{2}\mu\left(\pm\epsilon\dot{x}\right)W\end{aligned}$$

leading to the linear differential equation

$$\ddot{x} \pm \epsilon\left(\sqrt{2}\mu g\right)\dot{x} + \left(\frac{\sqrt{2}\mu\left[1 \pm \epsilon\left|R\dot{\theta}\right|\right]g}{\ell}\right)x = 0 \qquad (2)$$

The effect on the natural frequency is to replace fixed μ with its average effective value $\mu\left[1 \pm \epsilon\left|R\dot{\theta}\right|\right]$. The effect of a negative sign on ϵ is to provide a *negative damping* for the case of *dry friction.* This is confirmed by the observation that *the oscillation builds up* for most metal rods, in some cases sufficiently to finally eject the rod to one side or the other.

To deal with the control of autonomous oscillations, a **third approximation** introduces a non-linear friction characteristic, such as $\mu\left(1-\epsilon_1\left|\dot{x}_{\rm rel}\right|+\varepsilon_2\dot{x}_{\rm rel}^2\right)$, which has a negative slope (suggesting negative damping) only when $\left|\dot{x}_{\rm rel}\right| < \epsilon_1/2\varepsilon_2$



Since our initial investigation is restricted to small velocities $\left|\dot{x}\right| < \left|R\dot{\theta}\right|$

$$\begin{aligned}\left|\dot{x}_{\rm rel}\right| &= \left|R\dot{\theta}\right| - \dot{x} \text{ on the left-hand pulley}\\ \left|\dot{x}_{\rm rel}\right| &= \left|R\dot{\theta}\right| + \dot{x} \text{ on the right-hand pulley}\end{aligned}$$

$$\begin{aligned}F &= \sqrt{2}\mu\left\{\begin{array}{c}\left[1-\epsilon_1\left(\left|R\dot{\theta}\right|-\dot{x}\right)+\varepsilon_2\left(\left(R\dot{\theta}\right)^2-2\left|R\dot{\theta}\right|\dot{x}+\dot{x}^2\right)\right]\left(\frac{\ell-x}{2\ell}\right)\\ -\left[1-\epsilon_1\left(\left|R\dot{\theta}\right|+\dot{x}\right)+\varepsilon_2\left(\left(R\dot{\theta}\right)^2+2\left|R\dot{\theta}\right|\dot{x}+\dot{x}^2\right)\right]\left(\frac{\ell+x}{2\ell}\right)\end{array}\right\}W\\ &= \sqrt{2}\mu\left\{\begin{array}{c}-\left[1-\epsilon_1\left|R\dot{\theta}\right|+\varepsilon_2\left(\left(R\dot{\theta}\right)^2+\dot{x}^2\right)\right]\left(\frac{x}{\ell}\right)\\ -\left[-\epsilon_1\dot{x}+\varepsilon_2\left(2\left|R\dot{\theta}\right|\dot{x}\right)\right]\end{array}\right\}W\end{aligned}$$

leading to the non-linear differential equation

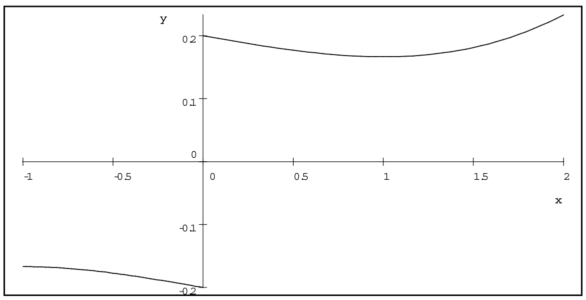
$$\frac{\ddot{x}}{\sqrt{2}\mu g}+\left[\left(-\epsilon_1+2\varepsilon_2\left|R\dot{\theta}\right|\right)+\frac{\varepsilon_2\dot{x}x}{\ell}\right]\dot{x}+\frac{1-\epsilon_1\left|R\dot{\theta}\right|+\varepsilon_2\left(R\dot{\theta}\right)^2}{\ell}x=0 \quad (3)$$

A. The effect on the natural frequency is to replace fixed μ with its average effective value $\mu\left[1-\epsilon_1\left|R\dot{\theta}\right|+\varepsilon_2\left(R\dot{\theta}\right)^2\right]$.

B. For infinitesimal amplitudes, the linearized damping coefficient $\sqrt{2}\mu g\left[\left(-\epsilon_1+2\varepsilon_2\left|R\dot{\theta}\right|\right)\right]$ is negative for the case $\left|R\dot{\theta}\right|<\epsilon_1/2\varepsilon_2$; at greater pulley velocities it is positive and there is a stable region near $x=0$; *evidently, the initial stability can be controlled by varying pulley speed!*

C. For larger amplitudes, we need to explore the non-linear term, which in this case does *not* lead to a stable limit cycle!

To deal with stable autonomous oscillations, a **fourth approximation** introduces a higher-order non-linear friction characteristic, such as $\mu\left(1-\epsilon_1\left|\dot{x}_{\text{rel}}\right|+\varepsilon_3\epsilon_1\left|\dot{x}_{\text{rel}}\right|^3\right)$, which has a negative slope (suggesting negative damping) when $\left|\dot{x}_{\text{rel}}\right|<\epsilon_1/3\varepsilon_3$



Since our initial investigation is restricted to small velocities $\left|\dot{x}\right|<\left|R\dot{\theta}\right|$

$$\begin{aligned}\left|\dot{x}_{\text{rel}}\right| &= \left|R\dot{\theta}\right|-\dot{x} \text{ on the left-hand pulley}\\ \left|\dot{x}_{\text{rel}}\right| &= \left|R\dot{\theta}\right|+\dot{x} \text{ on the right-hand pulley}\end{aligned}$$

$$\begin{aligned}F &= \sqrt{2}\mu\left\{\begin{array}{l}\left[1-\epsilon_1\left(\left|R\dot{\theta}\right|-\dot{x}\right)+\varepsilon_3\left(\left|R\dot{\theta}\right|^3-3\left(R\dot{\theta}\right)^2\dot{x}+3\left|R\dot{\theta}\right|\dot{x}-\dot{x}^3\right)\right]\left(\frac{\ell-x}{2\ell}\right)\\ -\left[1-\epsilon_1\left(\left|R\dot{\theta}\right|+\dot{x}\right)+\varepsilon_3\left(\left|R\dot{\theta}\right|^3+3\left(R\dot{\theta}\right)^2\dot{x}+3\left|R\dot{\theta}\right|\dot{x}+\dot{x}^3\right)\right]\left(\frac{\ell+x}{2\ell}\right)\end{array}\right\}W\\ &= \sqrt{2}\mu\left\{\begin{array}{l}-\left[1-\epsilon_1\left|R\dot{\theta}\right|+\varepsilon_3\left(\left|R\dot{\theta}\right|^3+3\left|R\dot{\theta}\right|\dot{x}^2\right)\right]\left(\frac{x}{\ell}\right)\\ -\left[-\epsilon_1\dot{x}+\varepsilon_3\left(3\left(R\dot{\theta}\right)^2\dot{x}+\dot{x}^3\right)\right]\end{array}\right\}W\end{aligned}$$

leading to a non-linear differential equation in the form

$$\ddot{x} + c\dot{x} + b\dot{x}^3 + a\dot{x}^2 x + \omega_o^2 x = 0 \tag{4}$$

where: the eigenvalue is a function of pulley velocity

$$\omega_o^2 = \frac{\sqrt{2}\mu g}{\ell}\left[1 - \epsilon_1 \left|R\dot{\theta}\right| + \varepsilon_3 \left(\left|R\dot{\theta}\right|^3 + 3\left|R\dot{\theta}\right| \dot{x}^2\right)\right]$$

and: the linear damping coefficient is negative for moderate pulley velocities

$$c = \sqrt{2}\mu g \left[-\epsilon_1 + 3\varepsilon_3 \left(R\dot{\theta}\right)^2\right]$$

and: the cubic damping coefficient is positive

$$b = \sqrt{2}\mu g \left[\varepsilon_3\right]$$

and: the remaining non-linear term has little effect on the averaged cycle (according to the KB-method), for moderate values of the coefficient

$$a = \frac{\sqrt{2}\mu g}{\ell}\left[3\varepsilon_3 \left|R\dot{\theta}\right|\right]$$

We can conclude that

1. Autonomous oscillations are initiated *iff* the linearized equation has negative damping

$$c = \left[-\epsilon_1 + 3\varepsilon_3 \left(R\dot{\theta}\right)^2\right] < 0$$

2. The rate at which the limit cycle is initially approached depends on

$$\frac{\dot{A}}{A} \propto -c = \sqrt{2}\mu g \left[\epsilon_1\right]$$

3. The limit-cycle amplitude may be estimated by the KB-method

$$A_\infty^2 = \frac{4}{3\omega_o^2}\left[\frac{-c}{b}\right] = \frac{4}{3\omega_o^2}\left[\frac{\epsilon_1}{\varepsilon_3} - 3\left(R\dot{\theta}\right)^2\right]$$

Extension of the initial heuristic study to static-friction lock-on: Dry friction ordinarily involves a friction-factor $\mu_{\max}$ which is much greater than the sliding friction μ. For simplicity, we accept the *first approximation* for the sliding friction, and describe the motion in a piecewise-linear fashion: when the rod is locked-on to the left pulley and moving to the right, the velocity is constant at

$$\dot{x} = \left| R\dot{\theta} \right| \tag{5a}$$

It will continue this motion until the weight transfer is sufficient to overcome the difference in static and dynamic friction

$$\sqrt{2}\mu_{\max}\left(\frac{\ell - x}{2\ell}\right)W < \sqrt{2}\mu\left(\frac{\ell + x}{2\ell}\right)W$$

which occurs when x passes the value

$$\frac{x_{crit}}{\ell} = \frac{\mu_{\max} - \mu}{\mu_{\max} + \mu} \tag{6}$$

From this point on, the rod obeys the vibrational Equation 1

$$\ddot{x} + \left(\frac{\sqrt{2}\mu g}{\ell}\right)x = 0$$

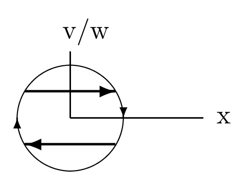
with the Initial Conditions

$$\begin{aligned} x_{ic} &= \left(\frac{\mu_{\max} - \mu}{\mu_{\max} + \mu}\right)\ell \\ \dot{x}_{ic} &= \left| R\dot{\theta} \right| \end{aligned}$$

and continues until the velocity reaches

$$\dot{x} = -\left| R\dot{\theta} \right| \tag{5b}$$

whereupon it locks-on in the opposite direction, and so on. This process can be plotted as a limit cycle on the phase-plane using only straight lines and circular arcs (if the vertical axis is normalized by the angular frequency ω_n of the fully-sliding Equation 1)



where heights of the straight lines represents the velocities of the pulleys $\pm R\dot{\theta}$ (and of the bar when it is locked-on to one or the other), the end of each straight line is at an x_{crit} which depends on $\mu_{\max}/\mu$ as shown in Equation 6, and connecting the straight lines with clockwise-motion along the circle represents the fully-sliding harmonic motion in between. The maximum amplitude is the radius of the circle

$$x_{\max} = \sqrt{\left(\frac{R\dot{\theta}}{\omega}\right)^2 + x_{crit}^2}$$

For this low-pulley-speed solution, the rod will fall off if $x_{\max} > \ell$.

If the pulleys are rotating steadily, there are three major regimes on the phase-plane:

- If the initial conditions are "small," the solution will describe a circle within the limit-cycle plotted above—an always-sliding solution as in Equation1.
- If the initial conditions are slightly larger, so that the solution-circle reaches the horizontal arrows in the plot above, the state will be captured by the arrow and continue on the limit cycle.
- If the magnitude of the initial velocity is greater than $\left|R\dot{\theta}\right|$, the velocity will be reduced by the friction on *both* pulleys, at a constant deceleration of $2\mu g$ until it reaches the horizontal arrows to be captured, either into the limit cycle or to be thrown-off the pulleys.

SUMMARY: If we start the pulleys rotating very slowly, the rod is likely to start out in the cycle shown in the phase-plane above; for high static friction and low dynamic friction, it is possible to throw the rod off the assembly when the rotational speed reaches a critical value.

If the pulleys are started suddenly (to break loose from static friction) and rotated briskly to maintain sliding, we may observe autonomous oscillations. Their steady-state amplitude can be explained by a friction characteristic which includes a small third-power dependence on rubbing speed.

Further investigation could be carried out using **numerical simulation**. To allow for both static friction and fourth-approximation dynamic friction, the governing equation needs to be formulated carefully.

We will call the positive-x side the right-hand side. At the instant of time numbered n, the support force R_n on the right–hand pulley, and the complementary support force $(W - R_n)$ on the left-hand pulley, are

$$\begin{aligned}(W - R_n) &= \left(\frac{\ell - x}{2\ell}\right) W + \frac{\left(r/\sqrt{2}\right)}{(2\ell)} \frac{W}{g} \ddot{x}_n \\ R_n &= \left(\frac{\ell + x}{2\ell}\right) W - \frac{\left(r/\sqrt{2}\right)}{(2\ell)} \frac{W}{g} \ddot{x}_n\end{aligned}$$

with the restriction that they must stay within the range $0 \leq R_n \leq W$. The simulation must examine the calculated support forces. If the physically possible range is exceeded, the value $R_n = W$ should be inserted for positive x_n, and $R_n = 0$ for negative x_n. For extreme cases, where the range of motion $-\ell \leq x_n \leq \ell$ is exceeded substantially, it may be appropriate to do a two-dimensional motion analysis.

Given a constant *inward* velocity of magnitude $\left|R\dot{\theta}\right|$ of each pulley surface, we obtain the speed of the bar *relative to* the **L**eft and **R**ight pulley surfaces

$$\begin{aligned}\dot{x}_{\mathrm{L}} &= \dot{x} - \left|R\dot{\theta}\right| \\ \dot{x}_{\mathrm{R}} &= \dot{x} + \left|R\dot{\theta}\right|\end{aligned}$$

and, in the absence of lock-on, the force/momentum balance is

$$\begin{aligned}\frac{W}{g}\ddot{x}_n = &-\mathrm{signum}\left(\dot{x}_{\mathrm{L}n}\right)\sqrt{2}\mu\left(1 - \epsilon_1\left|\dot{x}_{\mathrm{L}n}\right| \mp \varepsilon_2 \dot{x}_{\mathrm{L}n}^2 + \varepsilon_3\left|\dot{x}_{\mathrm{L}n}\right|^3\right)(W - R_n) \\ &-\mathrm{signum}\left(\dot{x}_{\mathrm{R}n}\right)\sqrt{2}\mu\left(1 - \epsilon_1\left|\dot{x}_{\mathrm{R}n}\right| \mp \varepsilon_2 \dot{x}_{\mathrm{R}n}^2 + \varepsilon_3\left|\dot{x}_{\mathrm{R}n}\right|^3\right)(R_n)\end{aligned}$$

where $\mathrm{signum}\left(\dot{x}_{\mathrm{L}n}\right) \triangleq \left(\dot{x}_{\mathrm{L}n}/\left|\dot{x}_{\mathrm{L}n}\right|\right)$.

If you attempt to substitute central-difference equivalents for the derivatives in the expressions above, you will find that the resulting numerical algorithm is practically unmanageable. It is easier to use Euler's method (and perhaps later expand it to a Runge-Kutta scheme) by substituting v for $\dot{x}$ and $\dot{v}$ for $\ddot{x}$. The resulting procedure has the form

$$\begin{aligned}x_{n+1} &\cong x_n + \Delta t \times v_n \\ v_{n+1} &\cong v_n + \Delta t \times \frac{g}{W}\mathfrak{F}(x_n, v_n)\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{F} \;=\; & -\text{signum}\left(\dot{x}_{\mathrm{L}n}\right)\sqrt{2}\mu\left(1-\epsilon_1\left|v_{\mathrm{L}n}\right| \mp \varepsilon_2 v_{\mathrm{L}n}^2+\varepsilon_3\left|v_{\mathrm{L}n}\right|^3\right)(W-R_n) \\
& -\text{signum}\left(\dot{x}_{\mathrm{R}n}\right)\sqrt{2}\mu\left(1-\epsilon_1\left|v_{\mathrm{R}n}\right| \mp \varepsilon_2 v_{\mathrm{R}n}^2+\varepsilon_3\left|v_{\mathrm{R}n}\right|^3\right)(R_n)
\end{aligned}$$

and

$$\begin{aligned}
v_{\mathrm{L}n} &= v_n - \left|R\dot{\theta}\right| \\
v_{\mathrm{R}n} &= v_n + \left|R\dot{\theta}\right|
\end{aligned}$$

and

$$R_n \approx \left(\frac{\ell + x_n}{2\ell}\right) W$$

If we include the last term in the geometrically more precise expression for R_n

$$R_n \cong \left(\frac{\ell + x_n}{2\ell}\right) W - \frac{\left(r/\sqrt{2}\right)}{(2\ell)}\frac{W}{g}\left(\frac{v_{n+1}-v_n}{\Delta t}\right)$$

we must either solve for v_{n+1} explicitly in the second equation of the Euler procedure, or else correct R_n iteratively, or else settle for including older values v_{n-1}

$$R_n \cong \left(\frac{\ell + x_n}{2\ell}\right) W - \frac{\left(r/\sqrt{2}\right)}{(2\ell)}\frac{W}{g}\left(\frac{v_n-v_{n-1}}{\Delta t}\right)$$

To consider the possibility of lock-on, the progress of the solution must be examined at each time step: if the velocity goes through a maximum or a minimum $\dot{v} = 0$, there will be lock-on if v_{L} or v_{R} is zero (on the other hand, if zero-values of relative velocity are passed with momentum, there will be no discernible lock-on effect). If lock-on does occur, we revert to a constant-velocity solution $v = \pm\left|R\dot{\theta}\right|$ until the weight transfer is sufficient for friction μ to overcome static-friction $\mu_{\max}$

$$\begin{aligned}
\text{for } v &= -\left|R\dot{\theta}\right| \text{ when } \sqrt{2}\mu_{\max}(R_n) < \sqrt{2}\mu(W-R_n) \\
\text{for } v &= +\left|R\dot{\theta}\right| \text{ when } \sqrt{2}\mu_{\max}(W-R_n) < \sqrt{2}\mu(R_n)
\end{aligned}$$

whereupon we return to our sliding-friction algorithm.